An Introduction to Lévy Processes with Applications in Finance
Antonis Papapantoleon

Presented by: Kevin Malenfant

October 8, 2008
Why?

- Describe the observed reality of financial markets in a more accurate way than models based on Brownian motion
  - Asset price processes have jumps or spikes
  - Empirical distribution of asset returns exhibits fat tails and skewness
  - In the ‘risk-neutral world, we observe that implied volatilities are constant neither cross strike nor across maturities

Lévy processes?

- Processes with independent and stationary increments are named Lévy processes after the French mathematician Paul Lévy (1886–1971)
Let \((\Omega, \mathcal{F}, \mathcal{F}, P)\) be a filtered probability space, where \(\mathcal{F} = \mathcal{F}_T\) and the filtration \(\mathcal{F} = (\mathcal{F})_{t \in [0,T]}\) satisfies the usual conditions.

**Definition 2.1**

A càdlàg, adapted, real valued stochastic process \(L = (L_t)_{t \geq 0}\) with \(L_0 = 0\) a.s. is called a Lévy process if the following conditions are satisfied:

\[(L1): \text{L has independent increments, i.e. } L_t - L_s \text{ is independent of } \mathcal{F}_s \text{ for any } 0 \leq s < t \leq T.\]

\[(L2): \text{L has stationary increments, i.e. for any } s, t \geq 0 \text{ the distribution of } L_{t+s} - L_t \text{ does not depend on } t.\]

\[(L3): \text{L is stochastically continuous, i.e for every } t \geq 0 \text{ and } \epsilon > 0:\]

\[
\lim_{s \to t} P(|L_t - L_s| > \epsilon) = 0.
\]
Infinitely divisible

Let $X$ be a real valued random variable, denote its characteristic function by $\varphi_X$ and its law by $P_X$, hence $\varphi_X(u) = \int_{\mathbb{R}} e^{iux} P_X(dx)$.

**Definition 4.1**

The law $P_X$ of a random variable $X$ is **infinitely divisible**, if for all $n \in \mathbb{N}$ there exist i.i.d. random variables $X^{(1/n)}_1, \ldots, X^{(1/n)}_n$ such that

\[
X^d = X^{(1/n)}_1 + \ldots + X^{(1/n)}_n.
\]
Alternatively, we can characterize an infinitely divisible random variable using its characteristic function.

**Definition 4.2**

The law of a random variable $X$ is **infinitely divisible**, if for all $n \in \mathbb{N}$, there exists a random variable $X^{(1/n)}$, such that

$$(4.3) \quad \varphi_X(u) = \left(\varphi_{X^{(1/n)}}(u)\right)^n$$
Example 4.3 (Normal distribution)

Using the second definition, we can easily see that the Normal distribution is infinitely divisible. Let $X \sim \text{Normal}(\mu, \sigma^2)$, then we have

$$
\varphi_X(u) = \exp[iu\mu - \frac{1}{2}u^2\sigma^2] \\
= \exp \left[ n \left( iu\frac{\mu}{n} - \frac{1}{2}u^2\frac{\sigma^2}{n} \right) \right] \\
= \left( \exp \left[ u\frac{\mu}{n} - \frac{1}{2}u^2\frac{\sigma^2}{n} \right] \right)^n \\
= \left( \varphi_{X^{(1/n)}}(u) \right)^n
$$

where $X^{(1/n)} \sim \text{Normal} \left( \frac{\mu}{n}, \frac{\sigma^2}{n} \right)$. 
Example 4.4 (Poisson distribution)

Following the same procedure, we can easily deduce that the Poisson distribution is infinitely divisible. Let $X \sim \text{Poisson}(\lambda)$, then we have

$$
\varphi_X(u) = \exp[\lambda(e^{iu} - 1)]
= \exp \left[ n \frac{\lambda}{n} (e^{iu} - 1) \right]
= \left( \exp \left[ \frac{\lambda}{n} (e^{iu} - 1) \right] \right)^n
= \left( \varphi_{X^{(1/n)}}(u) \right)^n
$$

where $X^{(1/n)} \sim \text{Poisson} \left( \frac{\lambda}{n} \right)$. 

Presented by: Kevin Malenfant  
Lévy Processes with Applications in Finance
Infinitely divisible

Examples
- compound Poisson distribution
- exponential
- $\Gamma$-distribution
- geometric
- negative binomial
- Cauchy distribution
- strictly stable distribution

Counter-examples
- uniform
- binomial
Theorem 4.6

The law \( P_X \) of a random variable \( X \) is infinitely divisible if and only if there exists a triplet \((b, c, \nu)\), with \( b \in \mathbb{R}, \ c \in \mathbb{R}_+ \) and a measure satisfying \( \nu(\{0\}) = 0 \) and \( \int_{\mathbb{R}} (1 \wedge |x|^2) \nu(dx) < \infty \), such that

\[
E[e^{iuX}] = \exp[ibu - \frac{u^2c}{2} + \int_{\mathbb{R}} (e^{iux} - 1 - iux1_{\{|x|<1\}}) \nu(dx)].
\]
Theorem 5.1

Consider a triplet \((b, c, \nu)\) where \(b \in \mathbb{R}\), \(c \in \mathbb{R}_+\) and \(\nu\) is a measure satisfying \(\nu(\{0\}) = 0\) and \(\int_{\mathbb{R}} (1 \wedge |x|^2) \nu(dx) < \infty\). Then, there exists a probability space \((\Omega, \mathcal{F}, P)\) on which four independent Lévy processes exist, \(L^{(1)}, L^{(2)}, L^{(3)}\) and \(L^{(4)}\), where \(L^{(1)}\) is a constant drift, \(L^{(2)}\) is a Brownian motion, \(L^{(3)}\) is a compound Poisson process and \(L^{(4)}\) is a square integrable (pure jump) martingale with an a.s. countable number of jumps on each finite time interval of magnitude less that 1. Taking \(L = L^{(1)} + L^{(2)} + L^{(3)} + L^{(4)}\), we have that there exists a probability space on which a Lévy process \(L = (L_t)_{t \geq 0}\) with characteristic exponent

\[
(5.1) \quad \psi(u) = iub - \frac{u^2c}{2} + \int_{\mathbb{R}} (e^{iux} - 1 - iux1_{\{|x|<1\}}) \nu(dx)
\]

for all \(u \in \mathbb{R}\), is defined.
Outline of Proof

We split the Lévy exponent (5.1) into four parts

\[ \psi = \psi^{(1)} + \psi^{(2)} + \psi^{(3)} + \psi^{(4)} \]

where

\[ \psi^{(1)}(u) = iub, \quad \psi^{(2)}(u) = \frac{u^2c}{2} \]

\[ \psi^{(3)}(u) = \int_{|x| \geq 1} (e^{iux} - 1) \nu(dx), \]

\[ \psi^{(4)}(u) = \int_{|x| < 1} (e^{iux} - 1 - iux) \nu(dx). \]
Outline of Proof (Continued)

- let $\triangle L^{(4)}$ denote the jumps of the Lévy process $L^{(4)}$
- let $\mu^{(4)}$ denote the random measure counting the jumps of $L^{(4)}$
- construct a compensated compound Poisson process

\[
L^{(4,\epsilon)}_t = \sum_{0 \leq s \leq t} \triangle L^{(4)}_s \mathbb{1}_{\{1 > |\triangle L^{(4)}_s| > \epsilon\}} - t \left( \int_{1 > |x| > \epsilon} x \nu(dx) \right)
\]

\[
= \int_0^t \int_{1 > |x| > \epsilon} x \mu^{(4)}(dx, ds) - t \left( \int_{1 > |x| > \epsilon} x \nu(dx) \right)
\]
Show that the jumps of \( L^{(4)} \) form a Poisson point process.

Get that the characteristic function of \( L^{(4,\epsilon)} \) is

\[
\psi^{(4,\epsilon)}(u) = \int_{\epsilon<|x|<1} (e^{iux} - 1 - iux) \nu(dx).
\]

There exists a Lévy process \( L^{(4)} \) which is a square integrable martingale and \( L^{(4,\epsilon)} \to L^{(4)} \) uniformly on \([0, T]\) as \( \epsilon \to 0^+ \) with Lévy exponent \( \psi^{(4)} \).

Therefore, we can decompose any Lévy process into four independent Lévy processes \( L = L^{(1)} + L^{(2)} + L^{(3)} + L^{(4)} \), i.e.

\[
L_t = bt + \sqrt{c} W_t + \int_0^t \int_{|x| \geq 1} x \mu^L(ds, dx) \\
+ \left( \int_0^t \int_{|x| < 1} x \mu^L(ds, dx) - t \int_{|x| < 1} x \nu(dx) \right)
\]
The Lévy measure

Lévy measure

- The Lévy measure $\nu$ is a measure on $\mathbb{R}$ that satisfies

$$\nu(\{0\}) = 0 \text{ and } \int_{\mathbb{R}} (1 \wedge |x|^2) u(dx) < \infty.$$  

- A Lévy measure has no mass at the origin, but singularities (i.e. infinitely many jumps) can occur around the origin (i.e. small jumps).

- Intuitively speaking, the Lévy measure describes the expected number of jumps of a certain height in a time interval of length 1.

Example

The Lévy measure of the Lévy jump-diffusion is $\nu(dx) = \lambda \cdot F(dx)$; from that we can deduce that the expected number of jumps, in a time interval of length 1, is $\lambda$ and the jump size is distributed according to $F$. 

Presented by: Kevin Malenfant

Lévy Processes with Applications in Finance
Distribution function of the Lévy measure of the Poisson process and the Density of the Lévy measure of a compound Poisson process with double-exponentially distributed jumps.
The density of the Lévy measure of an NIG and an \( \alpha \)-stable process.
Proposition 6.1

Let $L$ be a Lévy process with triplet $(b, c, \nu)$.

(1) If $\nu(\mathbb{R}) < \infty$ then almost all paths of $L$ have a finite number of jumps on every compact interval. In that case, the Lévy process has finite activity.

(2) If $\nu(\mathbb{R}) = \infty$ then almost all paths of $L$ have an infinite number of jumps on every compact interval. In that case, the Lévy process has infinite activity.
Proposition 6.2

Let \( L \) be a Lévy process with triplet \((b, c, \nu)\).

1. If \( c = 0 \) and \( \int_{|x| \leq 1} |x| \nu(dx) < \infty \) then almost all paths of \( L \) have finite variation.

2. If \( c \neq 0 \) or \( \int_{|x| \leq 1} |x| \nu(dx) = \infty \) then almost all paths of \( L \) have infinite variation.
The Lévy measure

The Lévy measure must integrate $|x|^2 \land 1$ (red); it has finite variation if it integrates $|x| \land 1$ (blue); it is finite if it integrates 1 (orange).

Moreover, the Lévy measure carries information about the finiteness of the moments of a Lévy process; this is extremely useful information in mathematical finance, for the existence of a martingale measure. The finiteness of the moments of a Lévy process is related to the finiteness of an integral over the Lévy measure (more precisely, the restriction of the Lévy measure to jumps larger than 1 in absolute value, i.e. big jumps).

Proposition 6.3. Let $L$ be a Lévy process with triplet $(b, c, \nu)$. Then

1. $L_t$ has finite $p$-th moment for $p \in \mathbb{R}^+$ if and only if
   \[
   \int |x|^p \nu(dx) < \infty.
   \]

2. $L_t$ has finite $p$-th exponential moment for $p \in \mathbb{R}$ if and only if
   \[
   \int e^{px} \nu(dx) < \infty.
   \]

Proof. The proof of these results can be found in Theorem 25.3 in Sato (1999). Actually, the conclusion of this theorem holds for the general class of submultiplicative functions (cf. Definition 25.1 in Sato 1999), which contains $\exp(px)$ and $|x|^p \lor 1$ as special cases. □

In order to gain some understanding of this result and because it blends beautifully with the Lévy-Itô decomposition, we will give a rough proof of the sufficiency for the second part (inspired by Kyprianou 2005).

Recall from the Lévy-Itô decomposition, that the characteristic exponent of a Lévy process was split into four independent parts, the third of which is a compound Poisson process with arrival rate $\lambda := \nu(\mathbb{R}\setminus(-1,1))$ and jump magnitude $F(dx) := \nu(dx)\nu(\mathbb{R}\setminus(-1,1))1_{\{|x|\geq 1\}}$. Finiteness of $\int e^{pL_t}$ implies...
Proposition 6.3

Let $L$ be a Lévy process with triplet $(b, c, u)$. Then

1. $L_t$ has finite $p$-th moment for $p \in \mathbb{R}_+$ ($\mathbb{E}|L_t|^p < \infty$) if and only if

$$\int_{|x| \geq 1} |x|^p \nu(dx) < \infty.$$ 

2. $L_t$ has finite $p$-th exponential moment for $p \in \mathbb{R}$ ($\mathbb{E}[e^{pL_t}] < \infty$) if and only if

$$\int_{|x| \geq 1} e^{px} \nu(dx) < \infty.$$
A Lévy process has first moment if the Lévy measure integrates $|x|$ for $|x| \geq 1$ (blue) and second moment if it integrates $x^2$ for $|x| \geq 1$ (orange).
Martingales and Lévy Processes

Proposition 10.1
Let $L = (L_t)_{t \geq 0}$ be a Lévy process with Lévy triplet $(b, c, \nu)$ and assume that $\mathbb{E}|L_t| < \infty$. $L$ is a martingale if and only if $b = 0$.

Proposition 10.2
Let $L = (L_t)_{t \geq 0}$ be a Lévy process with Lévy exponent $\psi$ and assume that $\mathbb{E}[e^{uL_t}] < \infty$, $u \in \mathbb{R}$. The process $M = (M_t)_{t \geq 0}$, defined as

$$M_t = \frac{e^{uL_t}}{e^{t\psi(u)}}$$

is a martingale.
(C1) Specifying a Lévy triplet
- Advantage: the characteristic function and the pathwise properties are known and allows the construction of a rich variety of models.
- Drawback: parameter estimation and simulation (in the infinite activity case) can be quite involved

(C2) Specifying an infinitely divisible random variable as the density of the increments at time scale 1 (i.e. $L_1$).
- Advantage: allows the easy estimation and simulation of the process.
- Drawback: the structure of the paths might be unknown

(C3) Time-changing Brownian motion with an independent increasing Lévy process.
- Advantage: allows for easy simulation
- Drawback: estimation might be quite difficult
Finite Activity: Simulating the Lévy jump-diffusion

\[ L_t = b t + \sigma W_t + \sum_{k=1}^{N_t} J_k \]

where \( N_t \sim \text{Poisson}(\lambda t) \) and \( J \sim F(dx) \), at fixed time points \( t_1, \ldots, t_n \).

- simulate a standard normal variate
- transform it into a normal variate with variance \( \sigma \Delta t \), where \( \Delta t = t_i - t_{i-1} \) (denoted \( G_i \))
- simulate a Poisson random variate with parameter \( \lambda \Delta t \)
- simulate the law of jump sizes \( J \), i.e. simulate \( F(dx) \)
- if the Poisson variate is larger than zero, add the value of the jump.

The discretized trajectory is

\[ L_{t_i} = b t_i + \sum_{j=1}^{i} G_j + \sum_{k=1}^{N_{t_i}} J_k. \]
Infinite Activity: Simulating normal inverse Gaussian (NIG) process with parameters $\sigma$, $\theta$, $\kappa$ at fixed time points $t_1, \ldots, t_n$.

- simulate $n$ independent inverse Gaussian variables $l_i$ with parameters $\lambda_i = \frac{(\Delta t)^2}{\kappa}$ and $\mu_i = \Delta t$ where $\Delta t = t_i - t_{i-1}$, $i = 1, \ldots, n$
- simulate $n$ standard normal variables $G_i$
- set $\Delta L_i = \theta l_i + \sigma \sqrt{t_i} G_i$

The discretized trajectory is

$$L_{t_t} = \sum_{j=1}^{i} \triangle L_j.$$
Infinite Activity: Simulating a variance gamma (VG) process with parameters $\sigma, \theta, \kappa$; at fixed time points $t_1, \ldots, t_n$.

- simulate $n$ independent gamma variables $\Gamma_i$ with parameter $\frac{\Delta t}{\kappa}$
- where
  \[ \Delta t = t_i - t_{i-1}, \ i = 1, \ldots, n \]
- set $\Gamma_i = \kappa \Gamma_i$
- simulate $n$ standard normal variables $G_i$
- set $\Delta L_i = \theta \Gamma_i + \sigma \sqrt{\Gamma_i} G_i$

The discretized trajectory is

\[ L_{t_t} = \sum_{j=1}^{i} \Delta L_j. \]
**Generalized Hyperbolic** (Eberlein and Prause 2002)

\[ L_1 \sim \text{GH}(\alpha, \beta, \delta, \mu, \lambda) \]

Density:

\[
f_{\text{GH}}(x) = c(\lambda, \alpha, \beta, \delta)(\delta^2 + (x - \mu)^2)^{(\lambda - \frac{1}{2})/2} \times K_{\lambda - \frac{1}{2}}(\alpha \sqrt{\delta^2 + (x - \mu)^2}) \exp(\beta(x - \mu)),
\]

where

\[
c(\lambda, \alpha, \beta, \delta) = \frac{(\alpha^2 - \beta^2)^{\lambda/2}}{\sqrt{2\pi} \alpha^{\lambda - \frac{1}{2}} K_{\lambda}(\delta \sqrt{\alpha^2 - \beta^2})}
\]

and \(K_{\lambda}\) denotes the Bessel function of the third kind with index \(\lambda\).
**Generalized Hyperbolic** (Eberlein and Prause 2002)

Parameters:

- $\alpha > 0$ determines the shape
- $0 \leq |\beta| < \alpha$ determines the skewness
- $\mu \in \mathbb{R}$ the location
- $\delta > 0$ is a scaling parameter
- $\lambda \in \mathbb{R}$ affects the heaviness of the tails
  - $\lambda - 1$ we get the hyperbolic distribution
  - $\lambda = -\frac{1}{2}$ we get the normal inverse Gaussian (NIG)
**Generalized Hyperbolic** (Eberlein and Prause 2002)

**Characteristic Function:**

\[ \varphi_{GH}(u) = e^{iu\mu} \left( \frac{\alpha^2 - \beta^2}{\alpha^2 - (\beta + iu)^2} \right)^{\frac{1}{2}} \frac{K_\lambda(\delta \sqrt{\alpha^2 - (\beta + iu)^2})}{K_\lambda(\delta \sqrt{\alpha^2 - \beta^2})}, \]

**First moment:**

\[ \mathbb{E}[L_1] = \mu + \frac{\beta \delta^2}{\zeta} \frac{K_{\lambda+1}(\zeta)}{K_\lambda(\zeta)} \]

**Second moment:**

\[ \text{Var}[L_1] = \frac{\delta^2}{\zeta} \frac{K_{\lambda+1}(\zeta)}{K_\lambda(\zeta)} + \frac{\beta^2 \delta^4}{\zeta^2} \left( \frac{K_{\lambda+2}(\zeta)}{K_\lambda(\zeta)} - \frac{K_{\lambda+1}^2(\zeta)}{K_{\lambda}^2(\zeta)} \right), \]

with \( \zeta = \delta \sqrt{\alpha^2 - \beta^2} \)

**Lévy triplet:** \((E[GH], 0, \nu^{GH})\)
Popular Models

**Normal Inverse Gaussian** (Barndorff-Nielsen 1997)  
(GH with $\lambda = -1/2$)

Density:

$$f_{NIG}(x) = \frac{\alpha}{\pi} \exp \left( \delta \sqrt{\alpha^2 - \beta^2} + \beta (x - \mu) \right) \frac{K_1(\alpha \delta \sqrt{1 + \left(\frac{x-\mu}{\delta}\right)^2})}{\sqrt{1 + \left(\frac{x-\mu}{\delta}\right)^2}},$$

Characteristic Function:

$$\varphi_{NIG} (u) = e^{iu\mu} \frac{\exp(\delta \sqrt{\alpha^2 - \beta^2})}{\exp(\delta \sqrt{\alpha^2 - (\beta + iu)^2})}.$$

First Moment:

$$E[L_1] = \mu + \frac{\beta \delta}{\sqrt{\alpha^2 - \beta^2}}$$

Second Moment:

$$\text{Var}[L_1] = \frac{\delta}{\sqrt{\alpha^2 - \beta^2}} + \frac{\beta^2 \delta}{\left(\sqrt{\alpha^2 - \beta^2}\right)^3}$$
Real-world measure
We model the asset price process as the exponential of a Lévy process

\[ S_t = S_0 \exp L_t, \quad 0 \leq t \leq T \]

where, \( L \) is the Lévy process whose infinitely divisible distribution has been estimated from the data set available for the asset.

Risk-neutral measure
We model the asset price process as the exponential of a Lévy process

\[ S_t = S_0 \exp L_t, \quad 0 \leq t \leq T \]

where, the Lévy process \( L \) has the triplet \((\bar{b}, \bar{c}, \bar{\nu})\) and has a finite first moment and exponential moment. \( L \) then has the canonical decomposition

\[ L_t = \bar{b} t + \sqrt{\bar{c}} \bar{W}_t + \int_0^t \int_{\mathbb{R}} x(\mu^L - \nu^L)(ds, dx) \]

with

\[ \bar{b} = r - \delta - \bar{c}^2 - \int_{\mathbb{R}} (e^x - 1 - x)\bar{\nu}dx \]
Three predominant methods:

- **Transform Methods**
  - Simple and fast
  - Exotic options cannot be handled easily

- **PIDE Methods**
  - Complex and Exotic options can be treated easily
  - Slower speed compared to transform methods and increased computational complexity when handling options on several assets

- **Monte Carlo Methods**
  - Options on several assets can be treated easily
  - Slow computational speed
Let $\mathbb{Q}$ be a risk-neutral measure and $q_T(s_T)$ be the risk-neutral density for the log price.

The call value is then the discounted expected value of the payoff under $\mathbb{Q}$,

$$
C_T(k) = e^{-rT} \mathbb{E}_\mathbb{Q} \left[ \left( e^{s_T} - e^k \right)_+ \right] = \int_k^\infty e^{-rT}(e^u - e^k)q_T(u)\,du
$$

Note: If $e^k \to 0$ we have $k \to -\infty$ and hence $C_T \to S_0$. So $C_T(k)$ is not square integrable!
To make $C_T(k)$ square-integrable Carr and Madan introduce a damping factor

$$c_T \equiv \exp(\alpha k)C_T(k)$$

Idea: $\alpha > 0$ causes $c_T(k)$ to decay as $k \to -\infty$

Carr and Madan found that if $\mathbb{E}[S_T^{\alpha+1}]$ then $c_T(k)$ is square integrable

Fourrier transform:

$$\psi_T(v) = \int_{-\infty}^{\infty} e^{ivk}c_T(k)dk$$

Reversing the transform and undamping:

$$C_T(k) = \frac{\exp(-\alpha k)}{2\pi} \int_{-\infty}^{\infty} e^{-ivk}\psi_T(v)dv$$

$$\psi_T(v) = \frac{e^{-rT} \phi_T(v - (\alpha + 1)i)}{\alpha^2 + \alpha - v^2 + i(2\alpha + 1)v}$$
Valuation of European options using Laplace transforms (Raible)
Assumptions:

(D1) Assume that $\varphi_{L_T}(z)$, the extended characteristic function of $L_T$, exists for all $z \in \mathbb{C}$ with $\Im z \in I_1 \supset [0, 1]$.

(D2) Assume that $P_{L_T}$, the distribution of $L_T$, is absolutely continuous w.r.t. the Lebesgue measure $\lambda$ with density $\rho$.

(P1) Consider a European-style payoff function $f(S_T)$ that is integrable.

(P2) Assume that $x \mapsto e^{-Rx} |f(e^{-x})|$ is bounded and integrable for all $R \in I_2 \subset \mathbb{R}$.

(B1) Assume that $I_1 \cap I_2 \neq \emptyset$. 

Presented by: Kevin Malenfant
Theorem B.1.

Let $F_1$ and $F_2$ be measurable complex-valued functions on the real line. If $|F_1(x)|$ is bounded and if $F_2(x)$ is absolutely integrable, then the convolution $F_1 \ast F_2$, defined by

$$F_1 \ast F_2(x) := \int_{\mathbb{R}} F_1(x - y)F_2(y)dy,$$

is a well-defined function on $\mathbb{R}$. $F_1 \ast F_2$ is bounded and uniformly continuous.
Theorem B.2.

Let $F_1$ and $F_2$ be measurable complex-valued functions on the real line. Let $z \in \mathbb{C}$ and $R := \Re(z)$. If

$$\int_{\mathbb{R}} e^{-Rx} |F_1(x)| \, dx < \infty \quad \text{and} \quad \int_{\mathbb{R}} e^{-Rx} |F_2(x)| \, dx < \infty,$$

and if $x \mapsto e^{-Rx} |F_1(x)|$ is bounded, then the convolution $F(x) := F_1 \ast F_2(x)$ exists and is continuous for all $x \in B$, and we have

$$\int_{\mathbb{R}} e^{-Rx} |F(x)| \, dx < \infty \quad \text{and} \quad \int_{\mathbb{R}} e^{-zx} F(x) \, dx = \int_{\mathbb{R}} e^{-zx} F_1(x) \, dx \cdot \int_{\mathbb{R}} e^{-zx} F_2(x) \, dx.$$
Theorem B.3.

Let $F$ be a measurable complex-valued function on the real line. Let $R \in \mathbb{R}$ such that

$$f(z) = \int_{\mathbb{R}} e^{-zx} F(x) \, dx \ (z \in \mathbb{C} \ \Re(z) = R),$$

with the integral converging absolutely for $z = R$. Let $x \in B$ such that the integral

$$\int_{R-i\infty}^{R+i\infty} e^{zx} f(z) \, dz$$

exists as a Cauchy principal value. Assume that $F$ is continuous at the point $x$. Then

$$F(x) = \frac{1}{2\pi i} \int_{R-i\infty}^{R+i\infty} e^{zx} f(z) \, dz,$$

where the integral is to be understood as the Cauchy principal value if the integrand is not absolutely integrable.
Definition 15.1
Let \( L_h(z) \) denote the bilateral Laplace transform of a function \( h \) at \( z \in \mathbb{C} \), i.e. let
\[
L_h(z) := \int_{\mathbb{R}} e^{-zx} h(x) dx.
\]
The value of a European style option with payoff $f(S_T)$

$$V = e^{-rT} \int_{-\infty}^{\infty} f(S_0 e^x) q(x) dx$$

Consider the modified payoff function $g(x) := f(e^{-x})$ and let $\zeta = -\ln(S_0)$

$$f(S_0 e^x) = f(e^{-\zeta} e^x) = f(e^{-(\zeta-x)}) = g(\zeta - x)$$

Rewriting we have

$$V = e^{-rT} \int_{-\infty}^{\infty} g(\zeta - x) q(x) dx = e^{-rT} (g \ast q)(\zeta).$$
Take an $R \in \mathbb{R}$ satisfying

$$\int_{-\infty}^{\infty} e^{-Rx} |g(x)| < \infty$$

where $e^{-Rx} |q(x)|$ is bounded and

$$\int_{-\infty}^{\infty} e^{-Rx} |q(x)| < \infty.$$

With $u \in \mathbb{R}$ and the value of the option written as a convolution we can then take the Laplace transform to get the product of Laplace transforms

$$\mathcal{L}_V(R + iu) = e^{-rT} \mathcal{L}_g(R + iu) \mathcal{L}_q(R + iu)$$

Finally we can take the inverse Laplace transform to get

$$V(\zeta) = \frac{e^{\zeta R - rT}}{2\pi} \int_{\mathbb{R}} e^{iu\zeta} \mathcal{L}_g(R + iu) \phi_{LT}(iR - u) du.$$
Theorem 15.2

Assume that the above mentioned conditions are in force and let $g(x) := f(e^{-x})$ denote the modified payoff function of an option with payoff $f(x)$ at time $T$. Choose an $R \in I_1 \cap I_2$. Letting $V(\zeta)$ denote the price of this option, as a function of $\zeta := -\log S_0$, we have

$$(15.1) \quad V(\zeta) = \frac{e^{\zeta R - rT}}{2\pi} \int_{\mathbb{R}} e^{iu\zeta} L_g(R + iu) \varphi_{LT}(iR - u) du,$$

whenever the integral on the r.h.s. it exists (at least as a Cauchy principal value).
European Call:

\[(15.3) \quad L_g(z) = \frac{1}{z(z + 1)}\]

for \( z \in \mathbb{C} \) with \( \Re z = R \in I_2 = (-\infty, -1) \).

European Put: \( z \in \mathbb{C} \) with \( \Re z = R \in I_2 = (0, \infty) \).

European Digital Call \((f(S_T) = 1_{\{S_T > K\}})\):

\[(15.4) \quad L_g(z) = -\frac{1}{z} \left( \frac{K}{S_0} \right)^z\]

for \( z \in \mathbb{C} \) with \( \Re z = R \in I_2 = (-\infty, 0) \).

European Digital Put \((f(S_T) = 1_{\{S_T < K\}})\):

\[(15.5) \quad L_g(z) = \frac{1}{z} \left( \frac{K}{S_0} \right)^z\]

for \( z \in \mathbb{C} \) with \( \Re z = R \in I_2 = (0, \infty) \).
Let us denote by $G(S_t, t)$ the time-$t$ price of a European option with payoff function $g$ on the asset $S_t$ the price is given by

$$G(S_t, t) = e^{-r(T-t)}E[g(S_T)] =: V_t, \ 0 \leq t \leq T$$

By arbitrage theory, we know that the discounted option price process must be a martingale under a martingale measure. Therefore, any decomposition of the price process as

$$e^{-rt}V_t = V_0 + M_t + A_t$$

where $M \in \mathcal{M}_{loc}$ and $A \in \mathcal{A}_{loc}$, must satisfy $A_t = 0$ for all $t \in [0, T]$. This condition yields the desired PIDE.
NOW, for notational but also computational convenience, we work with the driving process $L$ and not the asset price process $S$, hence we derive a PIDE involving $f(L_t, t) := G(S_t, t)$, or in other words

$$f(L_t, t) = e^{-r(T-t)}E[g(S_0e^{L_T})] = V_t, \ 0 \leq t \leq T.$$
Assume that $f \in C^{2,1}(\mathbb{R} \times [0, T])$, An application of Itô’s formula yields:

\[
\begin{align*}
\text{d}(e^{-rt} V_t) &= \text{d}(e^{-rt} f(L_{t-}, t)) \\
&= e^{-rt} \{-rf(L_{t-}, t) \text{d}t + \partial_2 f(L_{t-}, t) \text{d}t + \partial_1 f(L_{t-}, t) b \text{d}t} \\
&\quad + \partial_1 f(L_{t-}, t) \sqrt{c} \text{d}W + \int_{\mathbb{R}} \partial_1 f(L_{t-}, t) z(\mu^L - \nu^L)(\text{d}z, \text{d}t) \\
&\quad + \frac{1}{2} \partial^2_1 f(L_{t-}, t) c \text{d}t \\
&\quad + \int_{\mathbb{R}} (f(L_{t-} + z, t) - f(L_{t-}, t) - \partial_1 f(L_{t-}, t) z)(\mu^L - \nu^L)(\text{d}z, \text{d}t) \\
&\quad + \int_{\mathbb{R}} (f(L_{t-} + z, t) - f(L_{t-}, t) - \partial_1 f(L_{t-}, t) z) \nu(\text{d}z) \text{d}t
\end{align*}
\]
The bounded variation part vanishes identically. Hence, the price of the option satisfies the partial integro-differential equation

\[ 0 = -rf(x, t) + \partial_2 f(x, t) + \partial_1 f(x, t)b + \frac{c}{2} \partial_1^2 f(x, t) \]

\[ + \int_{\mathbb{R}} (f(x + z, t) - f(x, t) - \partial_1 f(x, t)z) \nu(dz), \]

for all \((x, t) \in \mathbb{R} \times (0, T)\), subject to the terminal condition

\[ f(x, T) = g(e^x). \]
Monte Carlo methods

The payoff of the call option with strike $K$ at the time of maturity $T$ is $g(S_T) = (S_T - K)^+$ and the price is provided by the discounted expected payoff under a risk-neutral measure, i.e.

$$C_T(S, K) = e^{-rT}E[(S_T - K)^+]$$.

Simulate the terminal value of asset price $S_T = S_0 \exp L_T$. Let $S_{T_k}$ for $k = 1, \ldots, N$ denote the simulated values; then, the option price $C_T(S, K)$ is estimated by the average of the prices for the simulated asset values, that is

$$\hat{C}_T(S, K) = e^{-rT} \sum_{k=1}^{N} (S_{T_k} - K)^+$$,

and by the Law of Large Numbers we have that

$$\hat{C}_T(S, K) \to C_T(S, K) \text{ as } N \to \infty.$$
Empirical Evidence

Densities of hyperbolic (red), NIG (blue) and hyperboloid distribution (left). Comparison of the GH (red) and Normal distributions (with equal mean and variance).
Empirical Evidence

Empirical distribution and Q-Q plot of EUR/USD daily log-returns with fitted GH (red).
Implied volatilities of EUR/USD options and calibrated NIG smile.