The Distribution of Stock Price Changes – Part I

When long-distance running in Berkeley met the empirical distribution of asset returns

In September of 1986, my associate Jerry Baesel and I attended a U.C. Berkeley Graduate School of Business Finance Seminar for academics and practitioners. One of the participants was Richard Grinold, a senior professor in the School. We discovered that we shared not only an enthusiasm for finance but also for long-distance running. During the lunch break we went for a ten mile hill run and chatted about finance.

I mentioned the surprising results of a Ph.D. thesis that, because I had left the faculty, I had supervised unofficially at the University of California Graduate School of Administration (now the Graduate School of Management). The Ph.D. candidate, Richard L. Kusper, had discovered what seemed to me to be a remarkable empirical fit, using a certain well known distribution, to the (pooled) logs of daily stock wealth relatives. Note that the daily wealth relative is the total return from holding the stock from one trading day’s close until the close of the next trading day, including the effects of dividends, stock splits and other corporate events. On a day \( i \) when there are no “other corporate events,” the wealth relative \( W_i \) is

\[
W_i = \frac{S_i (P_i + D_i)/P_{i-1}}{P_i} = \frac{S_i}{P_i} + D_i/P_{i-1}
\]

\( \log W_i = \log P_i - \log P_{i-1} \), the change in the logarithm of the daily price. When we talk about price changes being approximately lognormally distributed, i.e. the changes in the logs of daily stock prices being normally distributed, we really mean that \( W_i \) is approximately lognormally distributed. The empirical distribution of \( \log W_i \) is the subject of this article.

Kusper used the 468 Fortune 500 stocks, as given in the July 1962 issue, for which the CRSP database then had daily wealth relatives. This accounted for most of the largest cap liquid U.S. securities. During the study, the data grew to cover the period from July 2, 1962 through December 30, 1984. Due to delistings, mergers, and other events, the initial set of 468 companies shrank to about 280 by the end of 1984. Nonetheless, with over 1.95 million data points and 5,235 trading days, the data set covered a wide range of economic conditions and industries. I told Professor Grinold that Kusper’s choice of analytic function fitted the tails of the distribution particularly well, as far out as the data went, some 15 standard deviations. Since the thesis hadn’t been published and the information was of possible financial value to Princeton Newport Partners, I didn’t disclose Kusper’s method or fitting function.

A few years later Rich Grinold called to ask whether, if we were to include data from the crash of 1987, the results would still hold up. I said (perhaps too casually, but later you’ll see that Kusper’s process tends to incorporate extremes) that I thought it might but hadn’t done the work. Grinold sounded justifiably doubtful and that’s where it was left. Grinold went on to hold several top executive positions at BARRA and more recently wrote *Active Portfolio Management*, 1994 (second edition, 1999) and became Managing Director, Advanced Strategies and Research, at Barclays Global Investors, where
he oversees a multibillion dollar portfolio.

Motivation for Kusper’s work began at Princeton Newport Partners in the early to mid 1970s, where we were intensely developing and using extensions of the Black-Scholes model. A central issue was forecasting volatility, which is the problem of forecasting the standard deviation of a series of future log wealth relatives, hence related to, but more special than, the problem of forecasting the actual distribution of log \( W \), itself. We started by using “recent” historical volatility to predict future volatility. For each immediate future period of a specified length, we experimented with past periods of varying duration and varying weighting schemes.

As a first simple mechanical way to explore the volatility prediction problem, we used the \( n \) most recent past log wealth relatives, equally weighted, to predict the volatility of the next \( m \) wealth relatives. To reduce the work, we limited \( n \) and \( m \) to powers of 2, and then calculated and examined the resultant matrix of standard deviations of prediction errors. This methodology reappears in Kusper’s work.

As I recall, we settled on a geometrically weighted decay with multiplier \( \alpha \) typically in the 0.98 to 0.99 range, with the most recent day getting relative weight \( 1 \), then \( \alpha \) etc. Since there were a finite number of data points, we had to truncate the series after \( n \) data points, for some \( n \). We found that as good a method as any was to choose \( \alpha \) and \( n \) such that \( \alpha = 1 - 1/n \). In this case the relative weight of the included data was 

\[
(1 + \alpha + \ldots + \alpha^{n-1})(1 - \alpha) = 1 - \alpha^n \\
\approx 1 - e^{-1} \approx 0.63
\]

and of the excluded data 0.37. So most of the “weight” is in the most recent \( n \) past points, hence \( \alpha = 0.98 \) to 0.99 corresponds approximately to a lookback of 50 to 100 days. As a function of \( \alpha \), the prediction error had a “broad” minimum, meaning that it was not very sensitive to the precise value of \( \alpha \). It seemed as though longer lookbacks (\( \alpha \) values closer to 1) worked a little better for longer periods of forward prediction, and conversely shorter lookback periods seemed a little better for predicting shorter periods. That very long lookback periods tended to increase the standard deviation of the prediction error, despite reduced “measurement error,” may be due to the variability of the volatility over time. This has multiple causes, among them changes in the character or fortunes of the individual company, and in the volatility of industry, marketwide and other “factors.”

A very short lookback greatly reduced these problems but measurement error increased sharply, for a net increase in prediction error.

Our volatility work moved rapidly beyond this. We incorporated the daily high and low information I had first used in 1967 (see Wilmott, Sept. 2002, p.45), and the implied volatility information from actual market prices of the family of options or derivatives on a given stock. Some of this we found first ourselves and some came from the burgeoning academic literature on the subject.

If you keep betting the farm on situations where you’re almost sure to make a moderate profit … eventually you won’t have a farm

At the same time that I was interested in predicting volatility for use in derivative pricing I was also interested in the distribution itself, in particular in the extent to which the logs of wealth relatives deviated from the Black-Scholes assumption of normality, as such deviations might have a significant effect on the pricing of options and other derivatives. An enormous volume of academic work from the 1960s onward had been done on the problem of finding an analytic description, with much more work done subsequent to Kusper’s thesis. In particular it was well known that the data for daily log wealth relatives had fatter tails than the normal distribution. Among the proposed descriptions were the use of infinitely divisible distributions by Mandelbrot, mixtures of normals, distributions, and various mixes of continuous and jump distributions.

We knew from our own trading that the market tended to price far out of the money options substantially above their Black-Scholes model prices. One might think, as do some options traders who took Black-Scholes as exact, that there was free alpha to be had by regularly shorting large numbers of “overpriced” far out of the money options. Those who practiced this would sell tens or hundreds of thousands of dollars worth of near-term contracts at prices of, say, 1/2 or less. Usually these options would expire worthless and the profits would attract more money and more bettors. Once in a while, the short sellers would be caught, with some ruined.

This phenomenon reminds me of a similar well known one in the gambling world. William Feller, in his introduction to the Theory of Probability, vol. I, 3rd edn (fn p.346) tells of a man who paid for his vacations in Monte Carlo every year by gambling. In Feller’s example, the gam-